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ON THE STABILIZATION OF RELATIVE EQUILIBRIUM AND STEADY-STATE MOTION OF A MECHANICAL SYSTEM BY PARTIAL DISSIPATION FORCES

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We consider a linear canonical transformation [1] taking a gyroscopic system to a normal form. We show that the transformation coefficients may be chosen real. The transformation obtained is applied to the investigation of possible stabilization up to asymptotic stability of the relative equilibrium and the steady-state motion of a mechanical system. The stabilization of mechanical systems by controls $u_j(q_i, q_i^{\cdot})$ was studied in [2-4]. In this paper we pose the more special problem of seeking the conditions which must be satisfied by forces of partial dissipation in order that the relative equilibrium or the steady-state motion of a mechanical system can be stabilized by them up to asymptotic stability. We must remark that a stable mechanical system can be stabilized up to asymptotic stability by a force $u(q_1^{\cdot}, \ldots, q_n^{\cdot})$ of an arbitrary nature if and only if it is possible to stabilize this system by only one dissipative force [2].

1. Reduction of a gyroscopic system to normal form. Let the equations of motion of a linear gyroscopic system, whose position is described by the generalized coordinates $q_1, \ldots, q_n$, have the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^{\cdot}}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1.1}$$

Here L is a function of the form

$$L = \frac{1}{2}\sum_{i,j=1}^{n}(b_{ij}' q_i^{\cdot} q_j^{\cdot} + b_{i,n+j}' q_i q_j^{\cdot} + b_{n+i,n+j}' q_i q_j) \tag{1.2}$$

where the $b_{ij}'(i, j = 1, \ldots, 2n)$ are constant coefficients. Equation (1.1) can be written in the Hamiltonian form

$$q_i^{\cdot} = \partial H / \partial p_i, \qquad p_i^{\cdot} = -\partial H / \partial q_i \qquad (i = 1, \ldots, n) \tag{1.3}$$

$$H = \frac{1}{2}\sum_{i,j=1}^{n}(a_{ij} p_i p_j + a_{i,n+j} p_i q_j + a_{n+i,n+j} q_i q_j)$$

We assume that the quadratic form H is positive definite. In this case the roots of the characteristic equation of system (1.3),

$$\Delta(\lambda) = 0 \tag{1.4}$$

are all purely imaginary and the equilibrium position $q_1 = q_2 = \ldots = q_n = 0$ is stable [1]. Let

$$\pm\lambda_1^{(1)} i, \pm\lambda_1^{(2)} i, \ldots, \pm\lambda_1^{(n_1)} i; \pm\lambda_2^{(n_1+1)} i, \ldots, \pm\lambda_2^{(n_2)} i; \ldots,$$
$$\pm\lambda_k^{(n_{k-1}+1)} i, \ldots, \pm\lambda_k^{(n)} i$$

be k groups of roots of Eq. (1.4).

There exists [1] a linear canonical transformation

$$x_i = \sum_{j=1}^{n} (b_{ij}q_j + b_{i,\,n+j}p_j), \qquad y_i = \sum_{j=1}^{n} (b_{n+i,\,j}q_j + b_{n+i,\,n+j}p_j) \qquad (i = 1, \ldots, n) \tag{1.5}$$

taking Eqs. (1.3) to the normal form

$$x_i^\bullet = y_i, \qquad y_i^\bullet = -(\lambda_s^{(i)})^2 x_i \quad (i = 1, \ldots, n;\ s = 1, \ldots, k) \tag{1.6}$$

In the general case the coefficients of this transformation are complex. Let us find a transformation with real coefficients. Let E be the unit matrix; A, B be the matrices of coefficients of the variables $q_1, \ldots, q_n, p_1, \ldots, p_n$ in the right-hand sides of Eqs. (1.3), (1.5); Γ be a matrix of the form

$$\Gamma = \begin{Vmatrix} 0 & E \\ -E & 0 \end{Vmatrix}$$

z, u be $2n$-column-vectors; $b_1, \ldots, b_{2n}$ be the row-vectors of matrix B; (z, u) be a scalar product; Cz, C^2, C', $|C|$ be the product of the square matrix C by the column vector z, the square of matrix C, the transpose of matrix C, the determinant of matrix C, respectively; $\pm \lambda_s i$ $(s = 1, \ldots, k)$ be the characteristic index belonging to the sth group of Eq. (1.4).

The matrices A', $(A')^2$ have simple elementary divisors, because otherwise the equilibrium state would be unstable. Therefore, each of the systems

$$(A')^2 z = -\lambda_s^2 z \qquad (s = 1, \ldots, k) \tag{1.7}$$

has $2(n_s - n_{s-1})$ linearly independent solutions. We construct the following sequence of eigenvectors of matrix $(A')^2$:

$$z_s^{(2n_{s-1}+2i-1)} = \alpha_s^{(2i-1)} \Big\{ u_s^{(2i-1)} + \sum_{j=1}^{i-1} [-(u_s^{(2i-1)}, \Gamma z_s^{(2n_{s-1}+2j)}) \times$$
$$\times z_s^{(2n_{s-1}+2j-1)} + (u_s^{(2i-1)}, \Gamma z^{(2n_{s-1}+2j-1)})\, z^{(2n_{s-1}+2j)}] \Big\}$$

$$z^{(2n_{s-1}+2i)} = A' z^{(2n_{s-1}+2i-1)} \qquad (i = 1, \ldots, n_s - n_{s-1};\ s = 1, \ldots, k;\ n_0 = 0)$$

Here $u_s^{(2i-1)}$ is some solution of system (1.7), linearly independent of the vectors $z_s^{(2n_{s-1}+1)}, \ldots, z_s^{(2n_{s-1}+2i-2)}$, while the real corfficients $\alpha_s^{(2i-1)}$ are chosen such that $(z_s^{(2n_{s-1}+2i-1)}, \Gamma z_s^{(2n_{s-1}+2i)}) = 1$. We set

$$b_i = (z_s^{(2i-1)})', \qquad b_{n+i} = (z_s^{(2i)})' \qquad (i = n_{s-1} + 1, \ldots, n_s;\ \ s = 1, \ldots, k;\ \ n_0 = 0) \tag{1.8}$$

It can be verified that equalities (1.8) define a matrix B of a linear canonical transformation with real coefficients taking Eqs. (1.3) to the normal form (1.6).

2. Stabilization of the relative equilibrium of a mechanical system. We consider a mechanical system subject to holonomic steady-state constraints, whose position relative to a moving reference frame x_1, x_2, x_3 is determined by the generalized coordinates $q_1, \ldots, q_n$. Suppose that potential forces and dissipative forces, not explicitly dependent on time, with a function $F(q_1^\cdot, \ldots, q_n^\cdot)$ of rank $p < n$, i.e. the dissipation is not total, act on the system being considered. We assume that the system is in a relative equilibrium position $q_1 = \ldots = q_n = 0$ which we take as the unperturbed motion. If the transfer inertial forces admit of a force function not explicitly dependent on time, and if the projections of the instantaneous angular velocity onto the axes x_1, x_2, x_3 are constant, the equations of unperturbed motion in the first approximation can be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i^\cdot}\right) - \frac{\partial L}{\partial q_i} = \frac{\partial F}{\partial q_i^\cdot} \qquad (i = 1, \ldots, n) \tag{2.1}$$

where L has the form (1.2). At first we shall assume that the Hamiltonian H corresponding to L is a positive-definite quadratic form in $q_1, \ldots, q_n, p_1, \ldots, p_n$. We represent the function F in the form

$$F = -\frac{1}{2}(\varphi_1^2 + \ldots + \varphi_p^2), \quad \left(\varphi_r = \sum_{j=1}^{n} c_{rj} q_j^\cdot, \quad r = 1, \ldots, p\right) \tag{2.2}$$

In the normal variables found with the aid of the real canonical transformation in Sect. 1, Eq. (2.1) and the functions $F, \varphi_1, \ldots, \varphi_p$ take the form

$$x_i^\cdot = y_i + \sum_{j=1}^{p} d_{ij}\varphi_j, \qquad y_i^\cdot = -\lambda_s^2 x_i + \sum_{j=1}^{p} d_{n+i,j}\varphi_j \tag{2.3}$$
$$(i = n_{s-1} + 1, \ldots, n_s; \; s = 1, \ldots, k)$$

$$F = -\frac{1}{2}\sum_{i,j=1}^{n}(\alpha_{ij}x_i x_j + 2n_{ij}x_i y_j + \alpha_{n+i,n+j}y_i y_j), \qquad \varphi_r = \sum_{j=1}^{n}(c_{rj}x_j + c_{r,n+j}y_j)$$

$$\alpha_{ij} = \sum_{r=1}^{p} c_{ri}c_{rj}, \qquad \alpha_{n+i,n+j} = \sum_{r=1}^{p} c_{r,n+i}c_{r,n+j}, \qquad n_{ij} = \sum_{r=1}^{p} c_{ri}c_{r,n+j}$$

Let $F_1, \varphi_1^{(1)}, \ldots, \varphi_p^{(1)}$ be parts of functions $F, \varphi_1, \ldots, \varphi_p$, depending only on $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_1}$. To the form F_1 we apply successively a linear real transformation, the same one for the series of variables $x_1, \ldots, x_{n_1}$; $y_1, \ldots, y_{n_1}$, taking the form

$$f_1 = \sum_{i,j=1}^{n_1}(\alpha_{ij} + \lambda_1^2 \alpha_{n+i,n+j})x_i x_j$$

into a sum of squares, and an orthogonal transformation taking the skew-symmetric form

$$f_2 = \sum_{i,j=1}^{n_1}(n_{ij} - n_{ji})x_i y_j$$

into a canonical form [5]. In what follows we retain the old notation for the new coefficients α'_{ij}, n'_{ij}, d'_{ij} and for the new variables x'_i, y'_i. We note that the transformations being considered do not alter the form of Eqs. (2.3).

Theorem. For dissipative forces to stabilize in the first approximation a normal variable x_l up to asymptotic stability, it is necessary and sufficient that the coefficients of function F_1

$$F_1 = -\frac{1}{2}\sum_{i,j=1}^{s}(\alpha_{ij}x_i x_j + 2n_{ij}x_i y_j + \alpha_{n+i,n+j}y_i y_j) \tag{2.4}$$

$$\alpha_{ij} + \lambda_1^2 \alpha_{n+i,n+j} = \delta_{ij}, \qquad n_{rt} - n_{tr} = 0 \quad (r, t \neq 1, 2; 3, 4; \ldots)$$

satisfy the inequalities

$$l \leqslant s, \qquad \lambda_1^2 (n_{l,\,l\pm1} - n_{l\pm1,\,l})^2 - 1 \neq 0 \tag{2.5}$$

Proof. Necessity. If $l > s$, i.e. the variables x_l, y_l do not occur in function F_1, then $\varphi_1, \ldots, \varphi_p$ do not depend on these variables, therefore, Eqs. (2.3) admit of the nontrivial solution

$$x_l = C_l \cos\lambda_1 t + D_l \sin\lambda_1 t, \qquad x_i = y_i = 0 \quad (i \neq l)$$

Let $l \leqslant s$. We take $l = 1$. We consider the sum

$$v = \sum_{r=1}^{p} \left[(c_{r1} \mp c_{r,\,n+2}\lambda_1)^2 + \lambda_1^2 \left(\frac{c_{r2}}{\lambda_1} \pm c_{r,\,n+1} \right)^2 \right] =$$

$$= \sum_{i=1}^{2} (\alpha_{ii} + \lambda_1^2 \alpha_{n+i,\,n+i}) \mp 2\lambda_1 (n_{12} - n_{21})$$

If the equality $\lambda_1^2 (n_{12} - n_{21})^2 = 1$, has been fulfilled, then, according to (2.4), v vanishes, whence follows

$$c_{r1} = \pm \lambda_1 c_{r,\,n+2}, \qquad c_{r2} = \mp \lambda_1 c_{r,\,n+1} \qquad (r = 1, \ldots, p)$$

In this case Eqs. (2.3) admit of the nontrivial solution

$$x_1 = C_1 \cos\lambda_1 t + D_1 \sin\lambda_1 t, \qquad x_1^{\cdot} = y_1, \qquad x_2 = \pm y_1 / \lambda_1, \qquad y_2 = x_2^{\cdot}$$

$$x_i = y_i = 0 \qquad (i = 3, \ldots, n)$$

Consequently, the variable x_1 is not stabilized up to asymptotic stability.

Sufficiency. Since $dH / dt = F$, where F is negative-constant and H is positive-definite, it follows from the Barbashin-Krasovskii theorem [6, 7] that the motion tends asymptotically to those trajectories along which $F \equiv 0$. The equalities [6]

$$x_i^{\cdot} = y_i, \qquad y_i^{\cdot} = -\lambda_1^2 x_i \qquad (i = 1, \ldots, n_1) \tag{2.6}$$

$$\varphi_r^{(1)} = 0 \qquad (r = 1, \ldots, p) \tag{2.7}$$

are fulfilled on these trajectories. Substituting the solution of Eqs. (2.6),

$$x_i = C_i \cos\lambda_1 t + D_i \sin\lambda_1 t, \qquad y_i = -C_i \lambda_1 \sin\lambda_1 t + D_i \lambda_1 \cos\lambda_1 t$$

into equalities (2.7) and taking into account that the functions $\sin\lambda_1 t$ and $\cos\lambda_1 t$ are linearly independent, we obtain

$$v_r = \sum_{l=1}^{s} (c_{rl} C_l + c_{r,\,n+l} \lambda_1 D_l) = 0, \qquad w_r = \sum_{l=1}^{s} (c_{rl} D_l - c_{r,\,n+l} \lambda_1 C_l) = 0 \tag{2.8}$$

$$(r = 1, \ldots, p)$$

If $s = 1$, then $C_1 = D_1 = 0$. If $s \geqslant 2$, Eqs. (2.8) are equivalent to the equality

$$V = \sum_{r=1}^{p} (v_r^2 + w_r^2) = C_1^2 + D_1^2 + C_2^2 + D_2^2 + 2\lambda_1 (n_{12} - n_{21})(C_1 D_2 - C_2 D_1) +$$

$$+ V_1 (C_3, D_3, \ldots, C_s, D_s) \tag{2.9}$$

where V_1 is nonnegative. When condition (2.5) is fulfilled, the function $(V - V_1)$ is positive definite, therefore, equality (2.9) is satisfied only for $C_1 = D_1 = C_2 = D_2 = 0$, whence follows $x_1 = 0$. The assertion is proved.

For $n_1 = 1$ the asymptotic stability condition takes the form

$$a_{11} + \lambda_1^2 a_{n+1,\,n+1} \neq 0 \qquad (2.10)$$

From the theorem's proof it follows that inequalities (2.5) are the conditions for the absence of nontrivial trajectories of Eqs. (2.1), along which the equalities

$$\varphi_i' = \sum_{j=1}^{n} c_{ij} q_j = 0 \qquad (i = 1, \ldots, p)$$

are fulfilled. Expressing from these equations the last p generalized coordinates in terms of the remaining $m = n - p$ and substituting this expression into Eqs. (2.1), we go on to investigate the existence of nontrivial trajectories of equations of form (1.1) with the function

$$L_1 = L(q_i, q_i^{\cdot}, q_{m+j}(q_i), q^{\cdot}_{m+j}(q_j^{\cdot})) \qquad (i = 1, \ldots, m;\ j = 1, \ldots, n - m) \qquad (2.11)$$

along which are fulfilled certain linear equalities

$$\psi_r(q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_m^{\cdot}) = 0 \qquad (r = 1, \ldots, p) \qquad (2.12)$$

obtained from the last p equations of system (2.1). The theorem can be applied in this case too if we set

$$L = L_1, \quad F = -{}^1/_2(\psi_1^2 + \ldots + \psi_p^2).$$

For example, if $n = 2$, $F = (c_{11}q_1^{\cdot} + c_{12}q_2^{\cdot})^2$, then the function ψ_1 is

$$\psi_1 = b_{14}'(1 + c_{11}^2 / c_{12}^2) q_1^{\cdot} + (c_{11} / c_{12})(b_{44}' - b_{33}') q_1$$

since without loss of generality we can set

$$L = {}^1/_2[(q_1^{\cdot})^2 + (q_2^{\cdot})^2 + 2b_{14}' q_1 q_2^{\cdot} + b_{33}' q_1^2 + b_{44}' q_2^2]$$

Conditions (2.5) are not fulfilled only when $s = 0$, i.e. $\psi_1 \equiv 0$, whence follow $b_{14}' = 0$, $b_{44}' = b_{33}'$. Thus, the relative equilibrium of a mechanical system with two degrees of freedom, with $b_{14}' \neq 0$, can be stabilized up to asymptotic stability by any dissipation of rank $p = 1$.

Example 1. We consider a frame, rotating around a vertical line, with a mathematical pendulum attached to the frame's rotation axis by means of elastic springs, so that the vertical plane in which the pendulum is located and the pendulum's suspension point can accomplish, respectively, torsional and vertical oscillations. The kinetic energy T and the force function U are

$$T = {}^1/_2 m[(x^{\cdot})^2 + l^2(\varphi^{\cdot})^2 + 2l\sin\varphi x^{\cdot}\varphi^{\cdot} + l^2\sin^2\varphi\omega^2 + 2l^2\omega\sin^2\varphi\psi^{\cdot} + l^2\sin^2\varphi(\psi^{\cdot})^2]$$

$$U = -{}^1/_2 k_2 x^2 - {}^1/_2 k_3\psi^2 + mgl\cos\varphi + mgx$$

Here Ψ is the angle between the vertical plane and the plane of the frame, x is the displacement of the pendulum's suspension point from the end of the undeformed spring, φ is the pendulum's angle of deflection from the vertical, k_2, k_3 are the stiffness factors of the springs, m, l are the pendulum's mass and length, ω is the frame's anglular velocity of rotation.

As the unperturbed motion we take the solution

$$q_1 = q_2 = q_3 = 0 \qquad (2.13)$$

$$q_1 = l(\varphi - \varphi_0),\ q_2 = x - x_0,\ q_3 = l\psi\ (\cos\varphi_0 = g / (l\omega^2),\ x_0 = mg / k_2)$$

The equations of perturbed motion in the first approximation are written in form (2.1) with the functions L and F

$$L = {}^1/_2 m[(q_1^{\cdot})^2 + 2\sin\varphi_0 q_1^{\cdot} q_2^{\cdot} + (q_2^{\cdot})^2 + \sin^2\varphi_0(q_3^{\cdot})^2 + 2\omega\sin 2\varphi_0 q_1 q_3^{\cdot}] - {}^1/_2[(\omega^2 - g^2 / l^2\omega^2)q_1^2 + k_2 q_2^2 + (k_3 / l)q_3^2]; \qquad F = -{}^1/_2(c_{11}q_1^{\cdot} + c_{12}q_2^{\cdot} + c_{13}q_3^{\cdot})^2$$

The conditions for the positive definiteness of H are

$$k_2 > 0,\ k_3 > 0,\ \omega^2 - g^2 / l^2 \omega^2 > 0$$

If we take $k_2 / m = 48\,\mathrm{sec}^{-2}$, $k_3 / m = 11\,\mathrm{m}^2/\mathrm{sec}^2$, $\omega = 6\,\mathrm{sec}^{-1}$, $l = 0{,}54\,\mathrm{m}$, then the function H is

$$H = {}^1/_2\, m\,(4p_1^2 - 4\sqrt{3}p_1p_2 + 4p_2^2 + {}^4/_3 p_3^2 - 8\sqrt{3}p_3q_1 + 63q_1^2 + 48q_2^2 + 36q_3^2)$$

The roots of characteristic equation (1.4) are all distinct, therefore, an analysis of stability condition (2.10) leads to the conclusion that solution (2.13) is asymptotically stable when the nonequality

$$(c_{12}^2 + c_{13}^2)\,[c_{13}^2 + (c_{12} + 2\sqrt{3}c_{11})^2]\,[c_{13}^2 + (27c_{12} / \sqrt{3} - 16c_{11})^2] \neq 0$$

is fulfilled.

Example 2. We consider a rigid body moving in a central Newtonian force field in a drag-free medium. A material point of mass m is located inside the body. We assume that the center of mass O of the mechanical body – point system moves along an unperturbable circular orbit with angular velocity ω. Let C be the center of forces, $CX_1X_2X_3$ be a fixed reference frame, $Ox_1x_2x_3$ be an orbital coordinate system, $O_1y_1y_2y_3$ be a coordinate system with axes directed along the body's principal central axes of inertia. We take the angles ψ, ϑ, γ, respectively, as the generalized coordinates q_1, q_2, q_3 [9], defining the position of $O_1y_1y_2y_3$ relative to $Ox_1x_2x_3$, and the coordinates q_4, q_5 defining the position of the point relative to $O_1y_1y_2y_3$

$$y_i = f_i\,(q_4,\ q_5)\ \ (i = 1,\ 2,\ 3)$$

Let the equalities

$$f_{20} = f_{30} = (\partial f_1 / \partial q_4)_0 = (\partial f_1 / \partial q_5)_0 = 0$$

be fulfilled, where f_{i0} is the value of function $f\,(q_4, q_5)$ at $q_4 = q_5 = 0$. To ensure the asymptotic stability of the relative equilibrium $q_1 = \ldots = q_5 = 0$, which we take as the unperturbed motion, we introduce viscous friction with a dissipative function $F = -1/2\,[(q_4^{\cdot})^2 + (q_5^{\cdot})^2]$.

The equations of perturbed motion in the first approximation have form (2.1), where

$$\begin{gathered}
2L = A_1\,(q_1^{\cdot})^2 + (A_2 + m_1f_{10}^2)\,(q_2^{\cdot})^2 + (A_3 + m_1f_{10}^2)\,(q_3^{\cdot})^2 + \\
+ 2\omega\,(A_2 + A_1 - A_3)\,q_1q_2^{\cdot} + [(A_2 - A_3)q_1^2 + 4\,(A_1 - A_3 - m_1f_{10}^2)q_2^2 + \\
+ 3\,(A_1 - A_2 - m_1f_{10}^2)q_3^2]\omega^2 + m_1\{(a_4q_4^{\cdot} + a_5q_5^{\cdot})^2 + (b_4q_4^{\cdot} + b_5q_5^{\cdot})^2 + 2f_{10}[(a_4q_4^{\cdot} + \\
+ a_5q_5^{\cdot})q_3^{\cdot} - (b_4q_4^{\cdot} + b_5q_5^{\cdot})q_2^{\cdot}] + 3f_{10}\omega^2\,[(\partial^2 f_1/\partial q_4^2)_0q_4^2 + 2\,(\partial^2 f_1 / \partial q_4\partial q_5\partial)_0 \times \\
\times q_4q_5 + (\partial^2 f_1/ \partial q_5^2)_0q_5^2 + 8q_2\,(b_4q_4 + b_5q_5) - 6q_3\,(a_4q_4 + a_5q_5)] - (b_4q_4 + b_5q_5)^2\omega^2\} - \\
- k_4q_4^2 - k_5q_5^2
\end{gathered}$$

$$m_1 = mM / (m + M), \qquad a_i = (\partial f_2 / \partial q_i)_0, \qquad b_i = (\partial f_3 / \partial q_i)_0$$
$$(i = 4{,}5)$$

To investigate the asymptotic stability we examine the function L_1, introduced earlier by equality (2.11), obtained from the function L for $q_4 = q_4^{\cdot} = q_5 = q_5^{\cdot} = 0$. The functions ψ_1, ψ_2 from equalities (2.12) are

$$\psi_i = 4\omega b_{i+3}\,(l_{12} / B_2)q_2 + 3a_{i+3}\,(B_2 - B_1 - B_3)\omega^2q_3 - b_{i+3}\,(l_{12} / B_2)q_1^{\cdot}$$

$$B_i = A_i + m_1f_{10}^2 \quad (i = 2,\ 3), \qquad l_{12} = (A_3 - A_1 - A_2)\omega, \qquad B_1 = A_1$$

If the nonequality

$$B_1B_2(B_2-B_1)^2+(B_2-B_1)B_3[-12B_1(B_3-B_1)-3B_2(B_3-B_2)-3(B_3-B_1-B_2)^2]+4B_2^2(B_3-B_2)(B_3-B_1)\neq 0 \qquad (2.14)$$

is fulfilled, the roots of Eq. (1.4) corresponding to the Lagrangian L_1 are all distinct, and the asymptotic stability condition (2.10) can be brought to the form

$$l_{12}(B_2-B_1-B_3)(b_4^2+b_5^2)(a_4^2+a_5^2)\neq 0$$

If nonequality (2.14) is violated, $n_1=2$ and the asymptotic stability condition obtained from criterion (2.10) is

$$l_{12}(B_2-B_1-B_3)(b_4a_5-a_4b_5)\neq 0$$

3. Stabilization of the steady-state motion. We consider a mechanical system subject to holonomic steady-state constraints, whose position is determined by the generalized coordinates $q_1, \ldots, q_n$, where the last k coordinates are cyclic. We take it that the indices r, s vary from one to $(n-k)$, while the indices m, l from $(n-k+1)$ to n. Suppose that potential forces with a force function $U=U(q_r)$, dissipative forces with a dissipative function $\Phi=-{}^1/_2\,[(\dot q_{n-k+1})^2+\ldots+(\dot q_n)^2]$, and certain constant forces F_m act on the system being considered, such that the system admits of the motion

$$q_r=0, \qquad \dot q_m=\dot q_{m0}=\text{const} \qquad (3.1)$$

Let the kinetic energy T be

$$T=\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}\dot q_i\dot q_j$$

Solution (3.1) is asymptotically stable in the first approximation [10] if there do not exist nontrivial trajectories of equations of form (1.1) with the function

$$L=\delta^2\Big(\frac{1}{2}\sum_{r,s}a_{rs}\dot q_r\dot q_s+\frac{1}{2}\sum_{r,l}a_{rl}\dot q_r\dot q_{l0}+\sum_{m,l}a_{ml}\dot q_{m0}\dot q_{l0}+U\Big) \qquad (3.2)$$

along which the equalities

$$\varphi_m=\sum_s a_{ms}^{\circ}\dot q_s+\sum_l\Big[\sum_s(\partial a_{ml}/\partial q_s)_0\,q_s\Big]\dot q_{l0}=0 \qquad (3.3)$$

are fulfilled. Let us apply the transformation proposed in Sect. 2 to the function L and to the function

$$F=-\frac{1}{2}\sum_m\varphi_m^2$$

Then the conditions for the absence of such nontrivial trajectories, and, consequently, the asymptotic stability conditions, take the forms (2.5), (2.10).

Example 3. We consider a mechanical system which is a gyroscope in a gimbal suspension contained in a casing which is rigidly attached to a rod. The rod can rotate relative to a fixed point O_1. The gyroscope's center of gravity is located on the rod's axis at a point O. The rod, the casing and the gimbals are taken to be weightless. Let $O_1X_1X_2X_3$ be a fixed coordinate system with the O_1X_3-axis directed vertically upwards, $Ox_1x_2x_3$ be a system fixed on the rod, where the Ox_3-axis is directed along the rod from point O to the point O_1, while the axis of the outer gimbal is directed along Ox_1.

The rod's position is determined by angles α_1 and β_1, where β_1 is the angle between the Ox_3-axis and the projection of the rod onto the plane $O_1X_1X_3$, α_1 is the angle between this projection and the O_1X_3-axis. The gyroscope's position relative to $Ox_1x_2x_3$ is determined by angles α, β, γ [11]. The kinetic energy T and the force function U are

$$2T = Ml^2 [\cos^2 \beta_1 (\alpha_1^{\cdot})^2 + (\sin^2 \alpha_1 + \cos^2 \beta_1)(\beta_1^{\cdot})^2] + A_1 (\alpha^{\cdot} \cos \beta + \beta_1^{\cdot} \cos \beta + \alpha_1^{\cdot} \cos \beta_1 \sin \alpha_1 \sin \beta + \alpha_1^{\cdot} \sin \beta_1 \cos \alpha \sin \beta)^2 + A_1 (\beta^{\cdot} + \alpha_1^{\cdot} \cos \beta_1 \cos \alpha - \alpha_1^{\cdot} \sin \beta_1 \sin \alpha)^2 + A_3 [(\alpha^{\cdot} - \alpha_1^{\cdot} \sin \beta_1) \sin \beta + \gamma^{\cdot} - (\alpha_1^{\cdot} \cos \beta_1 \sin \alpha + \alpha_1^{\cdot} \sin \beta_1 \cos \alpha) \cos \beta]^2$$

$$U = -Mgl \cos \beta_1 \cos \alpha_1 - {}^1/_2\, k_1 (\alpha - \alpha_0)^2 - {}^1/_2\, k_2 (\beta - \beta_0)^2$$

where M is the gyroscope's mass, A_1, A_3 are, respectively, the equatorial and axial moments of inertia of the gyroscope, l is the distance O_1O, k_1, k_2 are the coefficients of elasticity of the springs fixing the position $\alpha = \alpha_0$, $\beta = \beta_0$. The coordinate γ is cyclic.

Let F_5 be a constant moment balancing the moment of the dissipative forces, namely, $-k\gamma^{\cdot}$ on the steady-state motion

$$\alpha_1 = \beta_1 = 0,\ \alpha = \alpha_0 = \pi / 2,\ \beta = \beta_0 = \pi / 4,\ \gamma^{\cdot} = \gamma_0^{\cdot}$$

In such a case, if for the perturbations we retain the notation of the original variables, the functions L and φ_5, defined by equalities (3.2) and (3.3), are

$$2L = Ml^2 [(\alpha_1^{\cdot})^2 + (\beta_1^{\cdot})^2] + {}^1/_2 A_1 (\alpha^{\cdot} + \beta_1^{\cdot} + \alpha_1^{\cdot})^2 + A_1 (\beta^{\cdot})^2 + {}^1/_2 A_3 (\alpha^{\cdot} + \beta_1^{\cdot} - \alpha_1^{\cdot})^2 + \sqrt{2}\, A_3 \gamma_0^{\cdot} \beta (\alpha_1^{\cdot} + \beta_1^{\cdot} + \alpha^{\cdot}) - Mgl\alpha_1^2 - Mgl\beta_1^2 - k_1\alpha^2 - k_2\beta^2 \tag{3.4}$$

$$\varphi_5 = \sqrt{2} / 2\, (\alpha^{\cdot} + \beta_1^{\cdot} - \alpha_1^{\cdot})$$

Setting $\alpha = \alpha_1 - \beta_1$ in expression (3.4), we obtain the function L_1

$$L_1 = {}^1/_2 [(Ml^2 + 2A_1)(\alpha_1^{\cdot})^2 + Ml^2 (\beta_1^{\cdot})^2 + A_1 (\beta^{\cdot})^2 + 2 \sqrt{2}\, A_3 \gamma_0^{\cdot} \beta \alpha_1^{\cdot} - (Mgl + k_1)\alpha_1^2 + 2k_1\alpha_1\beta_1 - (Mgl + k_1)\beta_1^2 - k_2\beta^2]$$

The function ψ_1 from equality (2.12) has the form

$$\psi = k_1 (A_1 + Ml^2) \beta_1 + [-k_1 (A_1 + Ml^2) + A_1 Mgl] \alpha_1 - \sqrt{2} / 2 A_3 \gamma_0^{\cdot} Ml^2 \beta^{\cdot}$$

If we assume that the roots $\pm \lambda_1 i$, $\pm \lambda_2 i$, $\pm \lambda_3 i$ of Eq. (1.4) corresponding to L_1 are all distinct, then the condition for asymptotic stability with respect to the normal variable x_i takes the form

$$A_1 Mgl - k_1 (A_1 + Ml^2) [1 + k_1 (Ml^2 \lambda_i^2 - Mgl - k_1)^{-1}] - A_3^2 Ml^2 (\gamma_0^{\cdot})^2 \lambda_i^2 (A_1 \lambda_i^2 - k_2)^{-1} \neq 0$$

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ON A GAME PROBLEM OF CONFLICTING CONTROL

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We consider a differential game of guidance – evasion whose solution we are required to find in the class of pure position strategies. It is shown that the introduction into this problem of information discrimination of the opponent essentially distorts the meaning of the original game problem. It is known [1–3] that a differential game of guidance–evasion has a saddle point in the class of pure position strategies if the right-hand side of the equation describing the system's dynamics satisfies the condition

$$\max_u \min_v s'f(t, x, u, v) = \min_v \max_u s'f(t, x, u, v)$$

where the maximum and minimum are computed over admissible values of u and v; s is an arbitrary n-dimensional vector, the prime denotes the transpose. However, if the stated condition is violated, then, in general, an equilibrium situation does not exist in the class of strategies. Here the game's outcome depends essentially on whether the players have information on the controls realized in the system. A typical situation is when the players do not have such information available to them; in this case an interesting problem is that of seeking the positional minimax and maximin pure strategies of the players. Below we use the results obtained in [5, 6, 9] to construct such strategies in one example of conflicting control.

1. The physical sense of the problem being investigated is the following. We have a material point moving in a horizontal plane. The motion of this point is controlled by two players who form controls which are two-dimensional vectors $u[t]$ and $v[t]$. The first player chooses the control $u[t]$, while the vector $v[t]$ is chosen by the second player, and the realizations of the controls satisfy the constraints

$$\|u[t]\| \leqslant \mu, \quad \|v[t]\| \leqslant \nu \tag{1.1}$$

Here and subsequently $\|x\|$ denotes the Euclidean norm of vector x. There is some free play in the control system. therefore, instead of the control force $w[t] = u[t] - v[t]$ a certain force $w_*[t] = u_*[t] - v_*[t]$, is applied to the point where the vectors